

Computable Axiomatizability of Elementary Classes

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Abstract

The goal of this paper is to generalise Alex Rennet’s proof of the non-axiomatizability of the class of pseudo-o-minimal structures. Rennet showed that if \mathcal{L} is an expansion of the language of ordered fields and \mathbb{K} is the class of pseudo-o-minimal \mathcal{L} -structures (\mathcal{L} -structures elementarily equivalent to an ultraproduct of o-minimal structures) then \mathbb{K} is not computably axiomatizable. We give a general version of this theorem, and apply it to several classes of topological structures.

1 Introduction

Given a class \mathbb{K} of \mathcal{L} -structures, we write $\text{Th}(\mathbb{K})$ for the first order theory of \mathbb{K} ; that is, the set of all \mathcal{L} -sentences that are true in every structure of \mathbb{K} . Recall that a class \mathbb{K} is called *elementary* when $\mathcal{M} \models \text{Th}(\mathbb{K})$ if and only if \mathcal{M} is an element of \mathbb{K} , and that this holds if and only if \mathbb{K} is closed under ultraproducts and ultraroots [8, Corollary 8.5.13]. We say that an elementary class \mathbb{K} is *computably axiomatizable* if there is a computable axiomatization of $\text{Th}(\mathbb{K})$. With this terminology, Rennet proved that the class of pseudo-o-minimal fields (fields which are elementarily equivalent to an ultraproduct of o-minimal structures) is not computably axiomatizable [12].

Rennet’s paper was motivated by a number of results, among them Ax’s proof [1] that the theory of finite fields is decidable, and hence that the class of pseudo-finite fields is computably axiomatizable. As with the class of finite fields in the language of rings, the class of o-minimal structures in a language with an ordering and an extra unary predicate is not elementary. For each $n \in \mathbb{N}$, let \mathcal{M}_n be a copy of the real numbers in this language, where the ordering is interpreted by the usual ordering and the unary predicate is interpreted as $\{0, 1, \dots, n\}$. It is easy to see that each \mathcal{M}_n is o-minimal, but that the ultraproduct has a copy of the natural numbers as a definable set; this is clearly not a finite union of points and intervals, and hence the ultraproduct is not o-minimal. Thus, the class of o-minimal structures is not closed under ultraproducts, and so is not elementary.

Multiple proposals were made for possible axiomatizations of the class of pseudo-o-minimal structures (see [4] and [13], for instance). However, Rennet showed that in the case where the language expands that of ordered fields, there is no computable axiomatization for the theory of o-minimality, and hence the class of pseudo-o-minimal structures is not computably axiomatizable.

In [5], Haskell and Macpherson developed the notion of C -minimality, a generalization of o-minimality obtained by replacing the binary ordering by a ternary relation. Haskell and Macpherson looked at another generalization of o-minimality in [6], P -minimality, which is defined so that P -minimal fields are p -adically closed, just as o-minimal fields are real closed. Given the similarities between these settings and o-minimality, they are both contexts in which it is natural to ask whether Rennet’s theorem applies.

In this paper, we adapt Rennet’s proof to give a more general theorem, which can then be applied to other classes, including those of C -minimal and P -minimal structures. Section 2 contains the preliminaries and proof of the generalized theorem, while Section 3 contains some examples, including those mentioned above.

2 Preliminaries and the Generalized Theorem

We state our generalization of Rennet's theorem in the context of first order topological structures, as introduced by Pillay in [11]:

Definition 1. Let \mathcal{A} be a structure in a language with a formula $B(x, \bar{y})$ (where x is a single variable and \bar{y} is a tuple) such that the set of A -subsets $\{B(x, \bar{a})^{\mathcal{A}} : \bar{a} \subseteq A\}$ is a basis for a topology on A . We say that such an \mathcal{A} is a *first-order topological structure*, or simply a topological structure. Note that for any $\mathcal{A}' \equiv \mathcal{A}$, (\mathcal{A}', B) is also a topological structure.

We extend this notion by saying a class \mathbb{K} of \mathcal{L} -structures is *uniformly topological* if there is a single formula B such that each $\mathcal{A} \in \mathbb{K}$ is a topological structure with a basis given by B .

Recall the notion of a provability relation which plays a fundamental role in the proof of Gödel's Second Incompleteness Theorem (see, for instance, [2]): if Γ is a computable list of sentences in the language of arithmetic then there exists a binary relation $\text{prov}(s, d)$ such that in the standard model of Peano Arithmetic, $\text{prov}(s, d)$ if and only if d is the code number of a sentence and s is the code number for a proof of that sentence from Γ .

Theorem 1. Fix any computable language \mathcal{L} containing a unary predicate N . Suppose \mathbb{K} is a uniformly topological class of \mathcal{L} -structures whose topology is given by the formula $B(x, \bar{y})$. Moreover, suppose that for each $\mathcal{A} \in \mathbb{K}$, discrete definable subsets of A are finite. Let Λ be any computable subset of $\text{Th}(\mathbb{K})$.

Fix distinguished \mathcal{L} -formulas α , μ , and \leq which define subsets of N^3 , N^3 , and N^2 , respectively, without parameters. Also fix \emptyset -definable constants $0, 1 \in N$. Let T be the \mathcal{L} -theory described below:

(I) $(N, \alpha, \mu, \leq, 0, 1)$ is a model of the relational theory of Peano Arithmetic, PA .

(II) N is discrete: that is, T contains the sentence

$$\forall x \in N \exists \bar{a} \forall y (y \in N \wedge B(y, \bar{a}) \rightarrow y = x).$$

(III) For each $\psi \in \Lambda$, T contains $\forall x \in N \psi^{\leq x}$, where $\psi^{\leq x}$ is the sentence ψ with any occurrence of $N(t)$ replaced by $N(t) \wedge t \leq x$.

If T is consistent then there is an \mathcal{L} -structure $\mathcal{R}_\Lambda^{\mathcal{L}}$ which satisfies Λ , but is not elementarily equivalent to an ultraproduct of structures in \mathbb{K} .

It follows that the class $\{\mathcal{M} : \mathcal{M} \models \text{Th}(\mathbb{K})\}$ is not computably axiomatizable, since given any potential axiomatization Λ , the structure $\mathcal{R}_\Lambda^{\mathcal{L}}$ obtained in the theorem satisfies Λ but not $\text{Th}(\mathbb{K})$.

Proof. Assume that T is consistent. In every model of T , the interpretation of N is a model of Peano Arithmetic, and so by Gödel's Second Incompleteness Theorem, $T + \neg \text{Con}(T)$ is also consistent. Thus, there exists a model \mathcal{A} of $T + \neg \text{Con}(T)$. In particular, if $\text{prov}(s, d)$ is the provability relation for T and c is the Gödel number for the statement $0 = 1$ then $\mathcal{A} \models \exists s \text{prov}(s, c)$; that is, there exists $a \in N$ with $\mathcal{A} \models \text{prov}(a, c)$.

Fix $x \in N$ with x sufficiently large to code the proof of c (among other conditions, $a \leq x$ and $c \leq x$) and consider the structure \mathcal{A}_x which is identical to \mathcal{A} except that N is replaced by the initial segment $\{n \in N_{\mathcal{A}} : n \leq x\}$. Since \mathcal{A} satisfies the axiom schema (III), \mathcal{A}_x satisfies Λ . By Theorem 2.7 of [9], since $N_{\mathcal{A}_x}$ is an initial segment of $N_{\mathcal{A}}$, a model of the relational theory of Peano Arithmetic, it is a Δ_0 -elementary substructure of $N_{\mathcal{A}}$. Thus, since a being a code for a proof of $0 = 1$ in T is a Δ_0 -property of $a \in N_{\mathcal{A}_x}$, we have $N_{\mathcal{A}_x} \models \exists s \text{prov}(s, c)$.

We claim that \mathcal{A}_x is the desired structure $\mathcal{R}_\Lambda^{\mathcal{L}}$. Suppose for contradiction that \mathcal{A}_x is elementarily equivalent to an ultraproduct of structures in \mathbb{K} :

$$\mathcal{A}_x \equiv \mathcal{A}' = \prod_{i \in I} \mathcal{A}_i / \mathcal{U}$$

where \mathcal{U} is a non-principal ultrafilter on I , and every \mathcal{A}_i is a structure in \mathbb{K} . Since property (II), that N is discrete, is described by a first order sentence, it also holds in \mathcal{A}' , and hence, by Los's Theorem, it also holds in \mathcal{U} -most of the \mathcal{A}_i . Since each $\mathcal{A}_i \in \mathbb{K}$ and each $N_{\mathcal{A}_i}$ is trivially definable, by assumption \mathcal{U} -most of the $N_{\mathcal{A}_i}$ are finite.

Then, since $\mathcal{N}_{\mathcal{A}_x}$ is an initial segment of a model of PA , so is $N_{\mathcal{A}'}$ and \mathcal{U} -most of the $N_{\mathcal{A}_i}$. But \mathcal{U} -most of the $N_{\mathcal{A}_i}$ are finite, so \mathcal{U} -most of the $N_{\mathcal{A}_i}$ are finite initial segments of a model of PA , and hence are isomorphic to a substructure of \mathbb{N} with universe $I_n = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}$. That is, \mathcal{U} -most $N_{\mathcal{A}_i}$ are isomorphic, for some n_i , to the structure

$$\mathbb{N}_{n_i} = (I_{n_i}, \{(x, y, z) \in I_{n_i} : x + y = z\}, \{(x, y, z) \in I_{n_i} : xy = z\}, \{(x, y) \in I_{n_i} : x \leq y\}).$$

Let $c' \in N_{\mathcal{A}'}$ be a code for $0 = 1$ and $\text{prov}(d, s)$ the provability relation for T . Since $N_{\mathcal{A}_x} \equiv N_{\mathcal{A}'}$, we have $N_{\mathcal{A}'} \models \exists s \text{ prov}(s, c')$. Choose an index i such that $N_{\mathcal{A}_i} \models \exists s \text{ prov}(s, c'_i)$ and $N_{\mathcal{A}_i}$ is isomorphic to some \mathbb{N}_{n_i} as above. Then, since $N_{\mathcal{A}_i} \cong \mathbb{N}_{n_i}$ is a Δ_0 -elementary substructure of \mathbb{N} , there exists $b \in \mathbb{N}$ such that $\mathbb{N} \models \text{prov}(b, c)$, where $c \in \mathbb{N}$ is the image of $c'_i \in N_{\mathcal{A}_i}$. Because of the interpretation of $\text{prov}(b, c)$ in the standard model \mathbb{N} , this b corresponds to an actual proof of $0 = 1$ in T . Hence T is inconsistent, contradicting our assumption, and so \mathcal{A}_x cannot be elementarily equivalent to an ultraproduct of structures in \mathbb{K} . \square

Remark 1. Note that the requirement of the predicate N being included in the language is merely a convenience. Any occurrence of N could be replaced by a distinguished formula in one variable and the proof would be unaffected.

3 Consequences

The examples below are all straightforward consequences of the theorem, which amount to choosing an appropriate class for \mathbb{K} and showing that the theory T from the theorem is consistent.

The first pair of examples, P -minimality and C -minimality, are variations of \mathcal{O} -minimality designed for valued fields. While more detailed descriptions can be found in [6] and [5], for our purposes we need only a single example of each to use in our construction of a model of T .

Fix a prime p . Then any rational number can be written in the form $p^n \frac{a}{b}$ where $n, a, b \in \mathbb{Z}$ and $p \nmid a, b$. We define a valuation $v_p : \mathbb{Q} \rightarrow \mathbb{Z}$ by $v_p(p^n \frac{a}{b}) = n$. With appropriate choices of language, the completion \mathbb{Q} with respect to the norm $|x| = p^{-v(x)}$ is an example of a P -minimal structure, denoted \mathbb{Q}_p . In Chapter III of [10], Koblitz shows that the metric completion of the algebraic closure of \mathbb{Q}_p , denoted Ω_p , is an algebraically closed valued field; it then follows from [5, Theorem C] that Ω_p is an example of a C -minimal structure.

Let K be one of the fields described in the previous paragraph. In both cases, the exponential function $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges on the set $p\mathcal{O} = \{px \in K : v(x) \geq 0\}$, and is bijective on this domain. Moreover, \mathbb{Q}_p and Ω_p continue to be examples of P -minimal and C -minimal structures when the language is expanded by adding a symbol for the exponential function restricted to $p\mathcal{O}$; see [3, Theorem B] for the P -minimal case and [7, Theorem 1.6] for the C -minimal case.

We create a model of Peano arithmetic in K as follows: take $N = \{p^{pn} : n \in \mathbb{N}\}$, and define $\{0_N, 1_N, \alpha, \mu, \leq\}$ via the natural bijection $p^{pn} \mapsto n$. Note that these sets will not be definable in K using the usual language for P -minimal or C -minimal fields, even after adding a symbol for the restricted exponential function. Clearly, N will be isomorphic to the usual interpretation of the natural numbers, and hence will be a model of Peano arithmetic. In the examples below, we simply need to show that this structure is definable in our chosen language; the additional factor of p in the exponent will be required to ensure that $\exp(x)$ is defined everywhere required.

Example 1. Let $\mathcal{L}_d = \{+, -, \cdot, 0, 1, \text{Div}, \{P_n\}_{n \in \mathbb{N}}\}$ be the language used in [6], let \mathcal{L} be any expansion of $\mathcal{L}_d \cup \{\exp, N\}$, and let \mathbb{K} be the class of P -minimal \mathcal{L} -structures in which \exp is interpreted as the restricted exponential. Then the class $\mathbb{K}' = \{\mathcal{A} : \mathcal{A} \models \text{Th}(\mathbb{K})\}$ is not computably axiomatizable.

Proof. Let Λ be a purported axiomatization of $\text{Th}(\mathbb{K})$, and note that each $\mathcal{A} \in \mathbb{K}$ has a topology with uniformly definable basis $B(x, c, d) = \{x \in A : \text{Div}(x - c, d) \wedge \neg(x - c = d)\}$. It follows from Lemma 4.3 of [6] that every discrete definable set in a P -minimal structure is finite.

To show T is consistent, we consider \mathbb{Q}_p with $N = \{p^{pn} : n \in \mathbb{N}\}$ and $\{0_N, 1_N, \alpha, \mu, \leq\}$ interpreted as described above. Clearly, 0_N and 1_N are \emptyset -definable, and $x \leq y$ is equivalent to $\text{Div}(x, y)$. Moreover, $\alpha(x, y, z)$ is defined by $xy = z$. It remains to show that $\mu(x, y, z)$ is definable in the language.

As noted above, the restricted exponential function on \mathbb{Q}_p is bijective, and hence the function $\ln(x)$ given by $\ln(x) = y$ when $\exp(y) = x$ is definable in \mathcal{L} for $v(x) \geq 1$. We can thus take $\mu(x, y, z)$ to be the set defined by

$$\exp\left(\frac{\ln(x)\ln(y)}{p^2 \ln(p)}\right) = z.$$

To turn this into an \mathcal{L} -structure \mathcal{A} , we simply use a trivial interpretation of every relation, function, and constant symbol not in $\mathcal{L}_d \cup \{\exp, N\}$. Conditions (I) and (II) for T are satisfied by choice of N . Condition (III) follows from the fact that every initial segment of N is finite: for all $x \in N$, \mathcal{A}_x is a definitional expansion of \mathbb{Q}_p (as an \mathcal{L}_d -structure), which means it is P -minimal, and hence satisfies Λ . Thus $\mathcal{A} \models T$, and so by Theorem 1, there is a model of Λ which is not an element of \mathbb{K}' . \square

Example 2. Let $\mathcal{L}_c = \{+, -, \cdot, 0, 1, C\}$ be the language of C -minimal fields described in [5], let \mathcal{L} be any proper expansion of $\mathcal{L}_c \cup \{\exp\}$, and let \mathbb{K} be the class of C -minimal \mathcal{L} -structures in which \exp is interpreted as the restricted exponential. Then the class $\mathbb{K}' = \{\mathcal{A} : \mathcal{A} \models \text{Th}(\mathbb{K})\}$ is not computably axiomatizable.

Proof. Let Λ be a purported axiomatization of $\text{Th}(\mathbb{K})$, and note that $B(x, b, c) = \{x : C(b; x, c)\}$ gives a uniformly definable basis for a topology on each $\mathcal{A} \in \mathbb{K}$. As noted in Lemma 2.4 of [5], discrete definable sets in C -minimal structures are finite.

To show T is consistent, consider Ω_p with $N = \{p^{pn} : n \in \mathbb{N}\}$ and $\{0_N, 1_N, \alpha, \mu, \leq\}$ interpreted as described above. Again, 0_N and 1_N are \emptyset -definable, and $x \leq y$ is equivalent to $\neg C(y; x, 1)$, where 1 here is the multiplicative identity in the field, not in the set N . The exponential function is again bijective, which means α and μ are definable by the same formulas as in the P -minimal case. Then we can form an \mathcal{L} -structure in the same way as before, and it will satisfy T for the same reasons described above. \square

For our final two examples, we look to Pillay's paper [11]. In section 3 of that paper, Pillay defines a dimension rank D_A for first order topological structures, which we will not repeat here. He notes that every stable first order topological structure has the discrete topology, and so Theorem 1 cannot be applied to stable structures. However, he introduces a different notion of stability for such structures, which can be used:

Definition 2. A first order topological structure \mathcal{A} is said to be *topologically totally transcendental*, or t.t.t., if it satisfies the following properties:

- (A) Every definable set $X \subseteq A$ is a boolean combination of definable open sets.
- (B) Every definable set $X \subseteq A$ has $d(X) < \infty$, where $d(X)$ is the maximum choice of d such that X can be written as a disjoint union of nonempty definable sets X_1, \dots, X_d with each X_i both closed and open in X .
- (C) A has dimension, meaning $D_A(A) < \infty$.
- (D) The topology on A is Hausdorff.

Moreover, \mathcal{A} is said to be *t-minimal* if \mathcal{A} is t.t.t. and $D_A(A) = d(A) = 1$.

In the case of an ordered structure, t -minimality is equivalent to o-minimality [11, Proposition 6.2]. However, the definition is less restrictive in general. Since the ordering on the reals is definable in the field language, $(\mathbb{R}, +, \cdot)$ with the usual topology is t -minimal, while the structure $(\mathbb{C}, +, \cdot, P)$ with the usual topology and P interpreted as a predicate for the positive reals is an example of a t.t.t. structure which is not t -minimal.

Example 3. Let \mathcal{L} be a proper expansion of $\mathcal{L}_{\text{tf}} = \{+, \cdot, 0, 1, B\}$, where $+$ and \cdot are binary function symbols and B is an n -ary relation symbol for some $n \geq 2$, and let \mathbb{K} be the class of t -minimal \mathcal{L} -structures in which $B(x, \bar{y})$ gives a basis for a topology. Then the class $\mathbb{K}' = \{\mathcal{A} : \mathcal{A} \models \text{Th}(\mathbb{K})\}$ is not computably axiomatizable.

Proof. Let Λ be a purported axiomatization of $\text{Th}(\mathbb{K})$, and suppose $\mathcal{A} \in \mathbb{K}$ with $N \subseteq \mathcal{A}$ discrete and definable. Since \mathcal{A} is Hausdorff, each point $a \in N$ is closed, and since N is discrete, each $a \in N$ is open in N . Thus, $|N| = d(N)$ is finite by condition (B), and so discrete definable subsets in each $\mathcal{A} \in \mathbb{K}$ are finite.

Consider the real numbers with the usual interpretation of $+$, \cdot , 0 , and 1 , and N as a predicate for the natural numbers. If I is the set of all open intervals with endpoints in \mathbb{R} , then $|I| = |\mathbb{R}|$, so there exists a bijection $f : \mathbb{R} \rightarrow I$; take $B(x_1, \dots, x_n)$ to be the relation $x_1 \in f(x_2)$. Taking a trivial interpretation of every function, relation, and constant symbol not in \mathcal{L}_{tf} gives an \mathcal{L} -structure \mathcal{A} , which we claim is a model of T .

For (I), take $0_N = 0$, $1_N = 1$, α and μ the graphs of $+$ and \cdot restricted to N , and $x \leq y$ iff $x, y \in N$ and $\exists z(x + z^2 = y)$. Clearly, this gives a model of Peano Arithmetic. Since $N \cap (a - 1, a + 1) = \{a\}$ for every $a \in N$, we have (II), that N is discrete. It remains to show that for any $x \in N$, the structure \mathcal{A}_x is t.t.t.

First, note that B gives the usual topology on \mathbb{R} , which is clearly Hausdorff, and thus we have condition (D) of t.t.t. Moreover, the definable sets in \mathcal{A}_x are precisely the same as those in $(\mathbb{R}, +, \cdot, 0, 1, \leq)$, and hence are finite unions of points and intervals: this gives conditions (A) and (B). Finally, any definable set $X \subseteq A$ without interior in A must be a finite union of points, in which case $D_A(X) = 0$, and so $D_A(A) = 1$. This is equivalent to condition (C) by Proposition 3.7 of [11]. Thus, \mathcal{A} satisfies condition (III), which means T is consistent and Theorem 1 can be applied. \square

Remark 2. As with N , the inclusion of B in the language is merely a convenience. Given a distinguished formula for B that satisfies the assumptions for the structure to be t.t.t., we could (with more difficulty) interpret the function and relation symbols in such a way that we obtain essentially the same model of T given above.

Example 4. Let \mathcal{L} be an expansion of $\{+, \cdot, 0, 1, B, N\}$, where $+$ and \cdot are binary function symbols and B is an n -ary relation symbol for some $n \geq 2$, and let \mathbb{K} be the class of t -minimal \mathcal{L} -structures in which $B(x, \bar{y})$ gives a basis for a topology. Then the class $\mathbb{K}' = \{\mathcal{A} : \mathcal{A} \models \text{Th}(\mathbb{K})\}$ is not computably axiomatizable.

Proof. In the previous example, we have already shown everything necessary except that the structure \mathcal{A}_x has $d(A) = 1$. But this is equivalent to saying that \mathbb{R} (with its usual topology) is connected, which is clearly true. \square

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